Rodrigue's formula :-

$$\boxed{P_n(x) = \frac{1}{2^n\ n!} \cdot \frac{d^n}{dx^n}(x^2-1)^n}$$

Put $n = 0, 1, 2, 3, 4$

$\therefore\ P_0(x) = 1 \qquad , \quad P_1(x) = x$

$P_2(x) = \dfrac{3x^2-1}{2} \qquad , \quad P_3(x) = \dfrac{5x^3-3x}{2}$

$P_4(x) = \dfrac{35x^4 - 30x^2 + 3}{8}$

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Legender Polynomail

legender differential equeation :-

$$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$

The solution is :- $\boxed{y(x) = C_1 P_n(x) + C_2 Q_n(x)}$

$P_n(x) = \sum^{n} (-1)^m (2n-2m)!\ x^{n-2m}$

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Recurrence Relations :-

$$P'_{n+1}(x) = x P'_n(x) + (n+1) P_n(x)$$

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

$$P_n(x) = \frac{x}{n} P'_n(x) - P'_{n-1}(x)$$

$$P_n(x) = \frac{1}{2n+1} P'_{n+1}(x) - \frac{1}{2n+1} P'_{n-1}(x)$$

$$P'_n(x) = \frac{n}{x^2-1} \left[x P_n(x) - P_{n-1}(x) \right]$$

* * * * * * *

Legender generating function :-

$$\left[1-2xt+t^2\right]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x) \quad ; \quad |x| \leq 1 \ , |t| < 1$$

Put $x = 1$:-

$$\therefore \left[1-2t+t^2\right]^{-\frac{1}{2}} = (1-t)^{-1} = \sum_{n=0}^{\infty} t^n P_n(1)$$

$$\therefore \boxed{P_n(1) = 1}$$

Put $x = -1$

$$\therefore P_n(-1) = (-1)^n$$

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⊛ Definite integral involving Legender Polynomail :-

⊛ $$\int_{-1}^{1} P_n(x) \cdot f(x)\, dx = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} (x^2-1)^n f^{(n)}(x)\, dx$$

⊛ $$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \begin{cases} 0 & ; \; m \neq n \\ \dfrac{2}{2n+1} & ; \; m = n \end{cases}$$

* * * * *

⊛ Expansion of f(x) interms of Legender Polynomial :-

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + \cdots + a_n P_n(x)$$

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx$$

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sheet ①

II Prove that

(i) $P_n(-x) = (-1)^n P_n(x)$

$$\because P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n$$

$$\therefore P_n(-x) = \frac{1}{2^n n!} \cdot \frac{d^n}{d(-x)^n} ((-x)^2-1)^n$$

$$= \frac{(-1)^n}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n = (-1)^n P_n(x) \quad \#$$

(ii) $P_n'(1) = \frac{n(n+1)}{2}$

$$\because (1-x^2) y''(x) - 2x y'(x) + n(n+1) y(x) = 0$$

$\because P_n(x)$ هو أحد حلول المعادلة

$$\therefore (1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$$

وبوضع $x=1$

$$\therefore -2 P_n'(1) + n(n+1) P_n(1) = 0$$

$$\therefore P_n'(1) = \frac{n(n+1)}{2} P_n(1) = \frac{n(n+1)}{2} \quad \#$$

[2] By using Rodrigue's formula Prove that :-

$$2^n n!\, P_{n+1}(x) = (2n+1) D^{n-1} (x^2-1)^n + 2n D^{n-1} (x^2-1)^n$$

$$\because P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$$

بوضع $n = n+1$

$$\therefore P_{n+1}(x) = \frac{1}{2\cdot 2^n \cdot (n+1)\, n!} D^n \cdot D\left[(x^2-1)^n\right]$$

$$= \frac{1}{2\cdot 2^n \cdot (n+1)\, n!} D^n \left[(n+1)(x^2-1)^n 2x\right]$$

$$\therefore 2^n n!\, P_{n+1}(x) = D^{n-1} \left[(x^2-1)^n + 2x^2 n (x^2-1)^{n-1}\right]$$

$$= D^{n-1} (x^2-1)^n + 2n D^{n-1} (x^2-1+1)(x^2-1)^{n-1}$$

$$= D^{n-1} (x^2-1)^n + 2n D^{n-1} (x^2-1)^n + 2n D^{n-1} (x^2-1)^{n-1}$$

$$= (2n+1) D^{n-1} (x^2-1)^n + 2n D^{n-1} (x^2-1)^{n-1} \longrightarrow (1)$$

$\because D^{n-1} (x^2-1)^{n-1} = 2^{n-1} (n-1)!\, P_{n-1}(x)$ into (1) بالتعويض في

$$\therefore 2^n n!\, P_{n+1}(x) = (2n+1) D^{n-1} (x^2-1)^n + 2n\, 2^{n-1} (n-1)\, P_{n-1}(x)$$

$$\therefore 2^n n! \left[P_{n+1}(x) - P_{n-1}(x)\right] = (2n+1) D^{n-1} (x^2-1)^n$$

$$P_{n+1}(x) - P_{n-1}(x) = \frac{2n+1}{2^n n!} D^{n-1} (x^2-1)^n$$

#

[3] show that :

[i] $\int_0^1 P_{2n}(x)\,dx = 0$

$= \frac{1}{2}\int_{-1}^{1} P_{2n}(x)\,dx = \frac{1}{2}\int_{-1}^{1}(1)\cdot P_{2n}(x)\,dx$

$= \frac{1}{2}\cdot\frac{(-1)^n}{2^n\, n!}\int_{-1}^{1}(x^2-1)D^n(1)\,dx = 0$ حيث $D^n(1)=0$ المشتقة النونية للواحد الصحيح تساوي صفر

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$\because P_n(x) = \frac{1}{2n+1}\left[P'_{n+1}(x) - P'_{n-1}(x)\right]$

$\therefore \int_0^1 P_{2n}(x)\,dx = \int_0^1 \frac{1}{4n+1}\left[P'_{2n+1}(x) - P'_{2n-1}(x)\right]dx$

$= \frac{1}{4n+1}\int_0^1\left[P'_{2n+1}(x) - P'_{2n-1}(x)\right]dx$

$= \frac{1}{4n+1}\left[P_{2n+1}(x) - P_{2n-1}(x)\right]_0^1$

$= \frac{1}{4n+1}\left[\overset{1}{P_{2n+1}(1)} - \overset{1}{P_{2n-1}(1)} - \overset{0}{P_{2n+1}(0)} + \overset{0}{P_{2n-1}(0)}\right]$

$= \frac{1}{4n+1}(0) = 0$ //

(ii) $\int_{-1}^{1} x (P_n(x))^2 \, dx = 0$

$$= \int_{-1}^{1} x \, P_n(x) \, P_n(x) \, dx$$

$$\because x P_n(x) = \frac{n+1}{2n+1} \left[P_{n+1}(x) + \frac{n}{2n+1} P_{n-1}(x) \right]$$

$$= \int_{-1}^{1} \left[\frac{n+1}{2n+1} P_{n+1}(x) \cdot P_n(x) + \frac{n(n+1)}{(2n+1)^2} P_{n-1}(x) \, P_n(x) \right] dx = 0$$

[4] Express each of the following functions in terms of Legender Polynomail :-

(i) $f(x) = ax^2 + bx + c$

$$f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$$

$$\because a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) \, P_n(x) \, dx$$

$$a_0 = \frac{1}{2} \int_{-1}^{1} (ax^2 + bx + c) P_0(x) \, dx = \frac{1}{2} \int_{-1}^{1} (ax^2 + bx + c) \, dx$$

$$a_1 = \frac{3}{2} \int_{-1}^{1} (ax^2 + bx + c) \, P_1(x) \, dx = \frac{3}{2} \int_{-1}^{1} (ax^2 + bx + c) \cdot x$$

$$a_2 = \frac{5}{2} \int_{-1}^{1} (ax^2 + bx + c) \, P_2(x) \, dx = \frac{5}{2} \int_{-1}^{1} (ax^2 + bx + c) \cdot \frac{3x^2 - 1}{2}$$

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[5] obtain the first three non-zero coefficients in the legender series representation of the function:-

$$f(x)=\begin{cases} 0 & \text{if } -1<x<0 \\ 1 & \text{if } 0<x<1 \end{cases}$$

$$f(x)=a_0 P_0(x)+a_1 P_1(x)+a_2 P_2(x)+\dots$$

$$a_n=\frac{2n+1}{2}\int_{-1}^{1} f(x)\,P_n(x)\,dx$$

$$\therefore a_0=\frac{1}{2}\int_{0}^{1} P_0(x)\,dx=\frac{1}{2}\int_{0}^{1} dx=\frac{1}{2}$$

$$a_1=\frac{3}{2}\int_{0}^{1} P_1(x)\,dx=\frac{3}{2}\int_{0}^{1} x\,dx=\frac{1}{4}x^2\Big|_0^1=\frac{3}{4}$$

$$a_2=\frac{5}{2}\int_{0}^{1} P_2(x)\,dx=\frac{5}{4}\int_{0}^{1}(3x^2-1)\,dx$$

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Exampl $\boxed{10}$:- $\Longrightarrow$ مهم جداً لأن الدكتور شرحه في المحاضرة والمعيد حله في السكشن وأكد عليه

Prove that :- (1) $P_n(1) = 1$ (2) $P_n(-1) = (-1)^n$

(3) $P_{2n+1}(0) = 0$ (4) $P_{2n}(0) = \dfrac{(-1)^n \, 2n!}{2^{2n} (n!)^2}$

Legender generating function :-

$$\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)^{-\frac{1}{2}} \longrightarrow (1)$$

Put $x = 1$ into (1)

$$\therefore \sum_{n=0}^{\infty} t^n P_n(1) = (1-2t+t^2)^{-\frac{1}{2}} = (1-t)^{-1} = 1+t+t^2+t^3+ \dots$$

$$P_0(1) + P_1(1)\,t + P_2(1)\,t^2 + P_3(1)\,t^3 + \dots = 1+t+t^2+t^3+ \dots$$

$\therefore P_0(1) = 1$, $P_2(1) = 1$, $P_1(1) = 1$ $\therefore P_n(1) = 1$ #

Put $x = -1$ into (1)

$$\therefore \sum_{n=0}^{\infty} P_n(-1)\, t^n = (1+2t+t^2)^{-\frac{1}{2}} = (1+t)^{-1} = 1-t+t^2-t^3+t^4 \dots$$

$$\therefore P_0(-1) + P_1(-1)\,t + P_2(-1)\,t^2 + P_3(-1)\,t^3 \dots = 1-t+t^2-t^3+t^4- \dots$$

$\therefore P_n(-1) = (-1)^n$

#

1 2! 3!

$= P_0(0) + P_1(0)t + P_2(0)t^2 + P_3(0)t^3 + \cdots$

$\therefore P_1(0) = P_3(0) = P_5(0) \cdots = 0$

~~$\therefore P_{2n+1}(x) =$~~ $\therefore P_{2n+1}(0) = 0$

$$P_2(0) = \frac{(-1)^n n!(2n)!}{2^{2n}(n!)^2}$$

$$(1+t^2)^{-\frac{1}{2}} = 1 + \frac{-\frac{1}{2}}{1}t^2 + \frac{-\frac{1}{2}\cdot\frac{-3}{2}}{2!}t^4 + \cdots + \frac{-\frac{1}{2}\left(\frac{-3}{2}\right)\left(\frac{-1}{2}-n+1\right)}{n!}t^{2n}$$

$$\therefore P_{2n}(0) = \frac{\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\cdots\left(\frac{-1}{2}-n+1\right)}{n!}$$

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put $x = 0$ into (1)

$$\therefore \sum^{\infty} P_n(0)t^n = (1+t^2)^{-\frac{1}{2}}$$